HW #10

17- a. $Y = 2X - 1$

1.-

$$f_X(x) = \begin{cases} 4(1-2x), & 0 \le x \le 0.5 \\ 0, & \text{otherwise} \end{cases}$$

2.-

$$h(y) = \frac{y+1}{2}$$

h is continuous from $B = (-1, 0)$ onto $A = (0, 0.5)$

3.-

$$h'(y) = \frac{1}{2}$$

4.- $f_Y(y) = f_X(h(y))\,|h'(y)|$

$$= f_X\left(\frac{y+1}{2}\right)\left(\frac{1}{2}\right) = 4\left(1 - 2\left(\frac{y+1}{2}\right)\right)\left(\frac{1}{2}\right) = 2(1-y-1)$$

$f_Y(y) = 2(1-y)$. for $-1 \le y \le 0$

b. $Y = 1 - 2X$

2. $h(y) = X = \frac{Y-1}{-2}$

$$h(y) = \frac{1-y}{2}$$

h is cont. from $B = (0,1)$ onto $A = (0, 0.5)$

3.- $h'(y) = -\frac{1}{2}$

4.- $f_Y(y) = f_X(h(y))|h'(y)|$

$$= f_X\left(\frac{1-y}{2}\right)\left|-\frac{1}{2}\right| = 4\left(1 - 2\left(\frac{1-y}{2}\right)\right)\left(\frac{1}{2}\right)$$

$$= 2(1 - 1 + y) = 2y \quad 0 \leq y \leq 1$$

c. $Y = X^2$

2.- $h(y) = \sqrt{y}$

h is cont from $B = (0, 1/4)$ to $A = (0, 0.5)$

3.- $h'(y) = \frac{1}{2\sqrt{y}}$

4.- $f_Y(y) = f_X\left(\frac{1}{2\sqrt{y}}\right)\left|\frac{1}{2\sqrt{y}}\right| = 4\left(1 - 2\left(\frac{1}{2\sqrt{y}}\right)\right)\left(\frac{1}{2\sqrt{y}}\right) = \frac{2}{\sqrt{y}}\left(1 - \frac{1}{\sqrt{y}}\right)$

$f_Y(y) = \frac{2(\sqrt{y} - 1)}{y}$ for $0 < y \leq 1/4$

18.-

a) $Y = 3X$

1.- $f(x) = \begin{cases} \frac{3}{2}x^2 & , \quad -1 \le x \le 1 \\ 0 & , \quad \text{otherwise} \end{cases}$

2.- $h(y) = \frac{y}{3}$

h is cont from $B = (-3,3)$ onto $A = (-1,1)$

3.- $h'(y) = \frac{1}{3}$

4.- $f_Y(y) = f_X(h(y))\,|h'(y)| = f_X\left(\frac{y}{3}\right)\left(\frac{1}{3}\right) = \frac{3}{2}\left(\frac{y}{3}\right)^2\left(\frac{1}{3}\right)$

$$f_Y(y) = \frac{y^2}{18} \quad \text{for } -3 \le y \le 3$$

b) Similar

c) Similar

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a) $F(x) = \begin{cases} 1 - \frac{1}{x}, & x > 1 \\ 0, & \text{otherwise} \end{cases}$

$Y = \frac{1}{X}$

1.- $f_x(x) = \frac{dF(x)}{dx} = \frac{1}{x^2}$, so

$f_x(x) = \begin{cases} \frac{1}{x^2}, & x > 1 \\ 0, & \text{otherwise} \end{cases}$

2.- $h(y) = \frac{1}{y}$

h is cont from $B = (0, 1)$ onto $A = (1, \infty)$

3.- $h'(y) = -\frac{1}{y^2}$

4.- $f_y(y) = f_x(h(y)) \left| -\frac{1}{y^2} \right| = f_x\left(\frac{1}{y}\right)\left(\frac{1}{y^2}\right) = \frac{1}{\left(\frac{1}{y}\right)^2} \cdot \frac{1}{y^2} = y^2 \cdot \frac{1}{y^2} = 1$

thus $f_y(y) = \begin{cases} 1 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

Y is uniform on $(0, 1)$

20. $Y = e^X$

1.- $f_X(x) = \begin{cases} 1 & \text{if} \quad 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

2- $h(y) = \ln(y)$

h is cont from $B = (1, e)$ onto $A = (0,1)$

3.- $h'(y) = \frac{1}{y}$

4.- $f_Y(y) = f_X(h(y))\,|h'(y)| = f_X(\ln(y))\left(\frac{1}{y}\right)$

$$= 1 \cdot \frac{1}{y}$$

$f_Y(y) = \begin{cases} \frac{1}{y} & \text{if} \quad 1 \le y \le e \\ 0 & \text{otherwise} \end{cases}$

Chp 8

6. $EX_0 = \int_0^1 (x)2(1-x)\,dx = 2\int_0^1 (x-x^2)\,dx = 2\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1$

$= 2\left[\frac{1}{2}-\frac{1}{3}\right] = \frac{2}{6} = 1/3.$

In order to use the weak law of large numbers, $V(x)$ must be finite, which it is.

hence $\bar{X}$ converges in Probability to $\mu = 1/3$.

2. $EX = \int_{-1}^{1} x\left(\frac{3}{2}\right)x^2\,dx = \frac{3}{2}\left[\frac{x^4}{4}\right]_{-1}^{1} = \frac{3}{2}\left[\frac{1}{4}-\frac{1}{4}\right] = 0.$

Analog to Prob. 1, $\bar{X}$ converges in Prob. to 0.

4. Since $EX = \lambda$, analog to the prev. Problems.

$\bar{X}$ converges in prob. to λ

6. $EX = \int_0^\theta \frac{x}{\theta}\,dx = \frac{1}{\theta}\left[\frac{x^2}{2}\right]_0^\theta = \frac{\theta^2}{2\theta} = \frac{\theta}{2}$

Similar to the previous Problems

$\bar{X}$ converges in prob. to $\frac{\theta}{2}$.

11- Let X be the fracture strength

X is $N(\mu, 0.4)$

Find

$P(|\bar{x}-\mu| \le 0.2) = P(-0.2 \le \bar{x} - \mu \le 0.2)$

by CLT $\approx P\left(\frac{-0.2}{\frac{0.4}{\sqrt{100}}} \le Z \le \frac{0.2}{\frac{0.4}{\sqrt{100}}}\right)$

$= P\left(\frac{-0.2}{0.04} \le Z \le \frac{0.2}{0.04}\right)$

$= P(-5 \le z \le 1) \approx \underline{1}$

12- Find n such that

$P(|\bar{x} - \mu| \le 0.2) = 0.95$

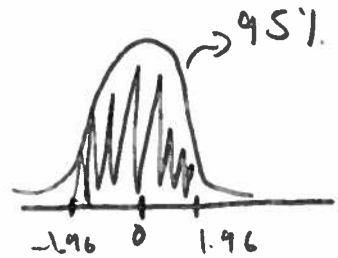
using CLT

$P\left(\frac{-0.2}{\frac{0.4}{\sqrt{n}}} \le Z \le \frac{0.2}{\frac{0.4}{\sqrt{n}}}\right) = 0.95$

we need to find Z_0 such that

$P(-Z_0 \le Z \le Z_0) = 0.95$

$Z_0 = 1.96$



then

$\frac{0.2}{\frac{0.4}{\sqrt{n}}} = 1.96$

$\Rightarrow \frac{\sqrt{n}}{0.5} = 1.96$, so $n = 0.96 \approx \underline{1}$

16- Let X = Carbon monoxide Concentration.

$$P(\bar{X} > 14) = ?$$

so

$$P(\bar{X} > 14) \overset{CLT}{\approx} P\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} > \frac{14-\mu}{\sigma/\sqrt{n}}\right)$$

$$= P\left(Z > \frac{14-12}{9/\sqrt{100}}\right)$$

$$= P\left(Z > \frac{(2)(10)}{9}\right) = P(Z > 2.22) = 0.01314$$

18- Let X = av. time to commute per day

a. $P(1.15 \le \bar{X} \le 1.25)$

by CLT $\approx P\left(\frac{1.15-1.2}{\frac{0.2}{\sqrt{36}}} \le Z \le \frac{1.25-1.2}{\frac{0.2}{\sqrt{36}}}\right)$

$$= P(-1.5 \le Z \le 1.5) = 0.8663$$

b) $P\left(\sum_{i=1}^{36} X_i \le 40\right) = P\left(\frac{\sum^{36} X_i}{36} \le \frac{40}{36}\right)$

$$= P(\bar{X} \le 1.11) \overset{CLT}{\approx} P\left(Z \le \frac{1.11-1.2}{\frac{0.2}{\sqrt{36}}}\right) =$$

$$= P(Z \le -2.7) = 0.0034$$

c) Observations be independent.

20 - $EX = \mu$ $\quad V(x) = \sigma^2 = 4$

Find μ such that

$$P\left(\sum_{i=1}^{50} X_i > 200\right) = 0.95$$

then

$$P\left(\sum_{i=1}^{50} X_i > 200\right) = P\left(\frac{\sum_{i=1}^{50} X_i}{50} > \frac{200}{50}\right) = P(\bar{X} > 4) \overset{CLT}{\approx} P\left(Z > \frac{4-\mu}{2/\sqrt{50}}\right) =$$

Find z_0 such that $P(Z > z_0) = 0.95$

$z_0 = -1.65$

Now

$$z_0 = \frac{4-\mu}{2/\sqrt{50}}$$

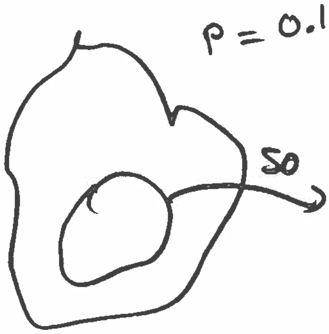
Solving for μ and subs $z_0 = -1.65$

$$\mu = -\frac{2}{\sqrt{50}}(-1.65) + 4 = \underline{4.46}$$

28- Let $X = \#$ of nonconformances in a Sample of size 50.

a.

$p = 0.10$



Find $P(X \le 5)$, X is Binomial $(50, 0.10)$

X can be approximated by the normal random variable W

with $\mu = np = (50)(0.10) = 5$ and $\sigma^2 = np(1-p) = 50(0.1)(0.9)$

but we also need to check that $p \pm 2\sqrt{\frac{p(1-p)}{n}}$ lies in $(0,1)$.

So $0.10 \pm 2\sqrt{\frac{(0.1)(0.9)}{50}} = 0.10 \pm 0.085$ lies in $(0,1)$.

then

$$P(X \le 5) \approx P(W \le 5.5) = P\left(Z \le \frac{5.5-5}{\sqrt{50(0.1)(0.9)}}\right)$$

$$= P(Z \le 0.23) = 0.59$$

b. Analog to a.

c. Analog to a.