CSCE 2100: Computing Foundations 1

Complete Induction

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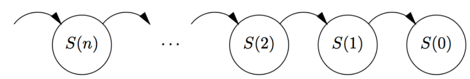
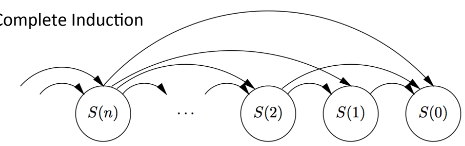
Complete Induction [1]

Weak Induction



Complete Induction



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Complete Induction [2]

Weak induction:

S(k) → S(k+1)

- It suffices that the statement is true for k to imply that it is true for k+1

Complete induction:

S(b),S(b+1),...,S(k) → S(k+1)

- We can use S(i) for any value between b (basis) and k.
- Multiple base cases may need to be proved

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Example 1: n = 2a + 3b [1]

1. Statement

S(n): For all integers n ≥0 there are integers a and b such that

n = 2a+ 3b.

2. Basis Cases

Basis

- n = 0
  pick a = 0 and b = 0 $\rightarrow 0 = 2 \times 0 + 3 \times 0$ ✓
- n = 1
  pick a = -1 and b = 1 $\rightarrow 1 = 2 \times (-1) + 3 \times 1$ ✓

The choices for a and b are not unique, but as longs as we find an example, we have shown that such integers exist.

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Example 1: n = 2a + 3b [2]

S(n): For all integers n ≥0 there are integers a and b such that n = 2a+ 3b.

3. Induction Hypothesis

We assume S(0),...,S(k) is true. In other words, for all integers k ≥ 1(largest basis case), there are integers a and b such that

k = 2a+ 3b.

4. "Target"

We need to show that S(0),...,S(k) implies S(k+1). In other words, there are integers a and b such that

k + 1 = 2a+ 3b.

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Example 1: n = 2a + 3b [3]

S(n): For all integers n ≥0 there are integers a and b such that n = 2a+ 3b.

5. Inductive Step

k ≥ 1 since 1 was the largest basis case

→ k-1 ≥ 0 → S(k-1) must be true.

k-1 = 2x + 3y (x and y are integers and exist by the induction hypothesis.)

+2 on both sides

k+1 = 2x + 3y + 2

k+1 = 2(x+1) + 3y

Choose a = x+1 and b = y. This proves the existence of two such integers and S(k+1) is true. ■

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Example 2: Postage [1]

1. Statement

- S(n): Show that any integer postage greater than 7 cents can be formed by using only 3-cent and 5-cent stamps.

2. Basis Cases

- We need 3 basis cases as we will see later…
  - n = 8: can be formed: 3+5
  - n = 9: can be formed: 3+3+3
  - n = 10: can be formed: 5+5

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Example 2: Postage [2]

- S(n): Show that any integer postage greater than 7 cents can be formed by using only 3-cent and 5-cent stamps.

3. Induction Hypothesis

- We assume that the statement is true for up to some k ≥ 10, i.e. for some integer postage **up to some** k ≥ 10, we can form it by using only 3-cent and 5-cent stamps.
  - We choose k ≥ 10 since it is the largest of the basis cases.

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Example 2: Postage [3]

- S(n): Show that any integer postage greater than 7 cents can be formed by using only 3-cent and 5-cent stamps.

4. "Target"

- What we need to show:
  We need to show that we can also form any integer postage for k+1 using only 3-cent and 5-cent stamps.

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Example 2: Postage [2]

- S(n): Show that any integer postage greater than 7 cents can be formed by using only 3-cent and 5-cent stamps.

5. Inductive Step

- Now we need to prove the statement for k+1
  k + 1 = (k-2) + 3
  - Since k ≥ 10, (k-2) must be ≥ 8 (So we are still within the range of the basis cases.).
  - By the induction hypothesis, we know that a postage of (k-2) can be formed using only 3-cent and 5-cent stamps.
  - We can add an additional 3-cent stamps to obtain a postage of (k+1). ■

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Summary Complete Induction

- Multiple basis cases might need to be proven
- Assume for the inductive hypothesis that S(i) is true for i = b,..,k whereby b is the size of largest basis case.
- Use any of the statements of the inductive hypothesis to prove S(k+1)
  - For complete induction we can refer to ANY of the statements S(b),…S(k), not just S(k) as in weak induction.

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